Quadratics

2.4	The quadratic function $x \mapsto ax^2 + bx + c$: its graph, y-intercept $(0, c)$. Axis of symmetry. The form $x \mapsto a(x-p)(x-q)$, x-intercepts $(p, 0)$ and $(q, 0)$. The form $x \mapsto a(x-h)^2 + k$, vertex (h, k).	Candidates are expected to be able to change from one form to another. Links to 2.3, transformations; 2.7, quadratic equations.	**Appl:** Chemistry 17.2 (equilibrium law). **Appl:** Physics 2.1 (kinematics). **Appl:** Physics 4.2 (simple harmonic motion). **Appl:** Physics 9.1 (HL only) (projectile motion).

Contents:

QUADRATICS

A **quadratic equation** is an equation of the form $ax^2 + bx + c = 0$ where a, b, and c are constants, $a \neq 0$.

A **quadratic function** is a function of the form $y = ax^2 + bx + c$, $a \neq 0$.

Quadratic functions are members of the family of **polynomials**. The first few members of this family are shown in the table.

Polynomial function	*Type*
$y = ax + b,\ \ a \neq 0$	linear
$y = ax^2 + bx + c,\ \ a \neq 0$	quadratic
$y = ax^3 + bx^2 + cx + d,\ \ a \neq 0$	cubic
$y = ax^4 + bx^3 + cx^2 + dx + e,\ \ a \neq 0$	quartic

A QUADRATIC EQUATIONS

SOLVING QUADRATIC EQUATIONS

To solve quadratic equations we have the following methods to choose from:

- **factorise** the quadratic and use the **Null Factor law**:

  If $ab = 0$ then $a = 0$ or $b = 0$.

- **complete the square**
- use the **quadratic formula**
- use **technology**.

The **roots** or **solutions** of $ax^2 + bx + c = 0$ are the values of x which satisfy the equation, or make it true.

For example, consider $x^2 - 3x + 2 = 0$.

When $x = 2$,
$$\begin{aligned} x^2 - 3x + 2 &= (2)^2 - 3(2) + 2 \\ &= 4 - 6 + 2 \\ &= 0 \quad \checkmark \end{aligned}$$

So, $x = 2$ is a root of the equation $x^2 - 3x + 2 = 0$.

SOLVING BY FACTORISATION

Step 1: If necessary, rearrange the equation so one side is zero.

Step 2: Fully factorise the other side.

Step 3: Use the Null Factor law: If $ab = 0$ then $a = 0$ or $b = 0$.

Step 4: Solve the resulting linear equations.

Caution: Do not be tempted to divide both sides by an expression involving x.
If you do this then you may lose one of the solutions.

For example, consider $x^2 = 5x$.

Correct solution

$$x^2 = 5x$$
$$\therefore\ x^2 - 5x = 0$$
$$\therefore\ x(x-5) = 0$$
$$\therefore\ x = 0 \text{ or } 5$$

Incorrect solution

$$x^2 = 5x$$
$$\therefore\ \frac{x^2}{x} = \frac{5x}{x}$$
$$\therefore\ x = 5$$

By dividing both sides by x, we lose the solution $x = 0$.

Example 1 Self Tutor

Solve for x: **a** $3x^2 + 5x = 0$ **b** $x^2 = 5x + 6$

Example 2 **Self Tutor**

Solve for x: **a** $4x^2 + 1 = 4x$ **b** $6x^2 = 11x + 10$

Example 3 **Self Tutor**

Solve for x: $3x + \frac{2}{x} = -7$

EXERCISE 1A.1

1 Solve the following by factorisation:

a $4x^2 + 7x = 0$
b $6x^2 + 2x = 0$
c $3x^2 - 7x = 0$
d $2x^2 - 11x = 0$
e $3x^2 = 8x$
f $9x = 6x^2$
g $x^2 - 5x + 6 = 0$
h $x^2 = 2x + 8$
i $x^2 + 21 = 10x$
j $9 + x^2 = 6x$
k $x^2 + x = 12$
l $x^2 + 8x = 33$

2 Solve the following by factorisation:

a $9x^2 - 12x + 4 = 0$
b $2x^2 - 13x - 7 = 0$
c $3x^2 = 16x + 12$
d $3x^2 + 5x = 2$
e $2x^2 + 3 = 5x$
f $3x^2 + 8x + 4 = 0$
g $3x^2 = 10x + 8$
h $4x^2 + 4x = 3$
i $4x^2 = 11x + 3$
j $12x^2 = 11x + 15$
k $7x^2 + 6x = 1$
l $15x^2 + 2x = 56$

3 Solve for x:

a $(x+1)^2 = 2x^2 - 5x + 11$
b $(x+2)(1-x) = -4$
c $5 - 4x^2 = 3(2x+1) + 2$
d $x + \frac{2}{x} = 3$
e $2x - \frac{1}{x} = -1$
f $\frac{x+3}{1-x} = -\frac{9}{x}$

EXERCISE 1A.1

1 a $x = 0, -\frac{7}{4}$ b $x = 0, -\frac{1}{3}$ c $x = 0, \frac{7}{3}$
d $x = 0, \frac{11}{2}$ e $x = 0, \frac{8}{3}$ f $x = 0, \frac{3}{2}$
g $x = 3, 2$ h $x = 4, -2$ i $x = 3, 7$
j $x = 3$ k $x = -4, 3$ l $x = -11, 3$

2 a $x = \frac{2}{3}$ b $x = -\frac{1}{2}, 7$ c $x = -\frac{2}{3}, 6$
d $x = \frac{1}{3}, -2$ e $x = \frac{3}{2}, 1$ f $x = -\frac{2}{3}, -2$
g $x = -\frac{2}{3}, 4$ h $x = \frac{1}{2}, -\frac{3}{2}$ i $x = -\frac{1}{4}, 3$
j $x = -\frac{3}{4}, \frac{5}{3}$ k $x = \frac{1}{7}, -1$ l $x = -2, \frac{28}{15}$

3 a $x = 2, 5$ b $x = -3, 2$ c $x = 0, -\frac{3}{2}$
d $x = 1, 2$ e $x = \frac{1}{2}, -1$ f $x = 3$

SOLVING BY 'COMPLETING THE SQUARE'

As you would be aware by now, not all quadratics factorise easily. For example, x^2+4x+1 cannot be factorised by simple factorisation. In other words, we cannot write x^2+4x+1 in the form $(x-a)(x-b)$ where a, b are rational.

An alternative way to solve equations like $x^2+4x+1=0$ is by 'completing the square'.

Equations of the form $ax^2+bx+c=0$ can be converted to the form $(x+p)^2=q$ from which the solutions are easy to obtain.

Example 4 **Self Tutor**

Solve exactly for x: **a** $(x+2)^2 = 7$ **b** $(x-1)^2 = -5$

Example 5 Self Tu

Solve for exact values of x: $x^2 + 4x + 1 = 0$

If the coefficient of x^2 is not 1, we first divide throughout to make it 1.

Example 6 **Self Tutor**

Solve exactly for x: $-3x^2 + 12x + 5 = 0$

EXERCISE 1A.2

1 Solve exactly for x:

a $(x+5)^2=2$
b $(x+6)^2=-11$
c $(x-4)^2=8$
d $(x-8)^2=7$
e $2(x+3)^2=10$
f $3(x-2)^2=18$
g $(x+1)^2+1=11$
h $(2x+1)^2=3$
i $(1-3x)^2-7=0$

2 Solve exactly by completing the square:

a $x^2-4x+1=0$
b $x^2+6x+2=0$
c $x^2-14x+46=0$
d $x^2=4x+3$
e $x^2+6x+7=0$
f $x^2=2x+6$
g $x^2+6x=2$
h $x^2+10=8x$
i $x^2+6x=-11$

3 Solve exactly by completing the square:

a $2x^2+4x+1=0$
b $2x^2-10x+3=0$
c $3x^2+12x+5=0$
d $3x^2=6x+4$
e $5x^2-15x+2=0$
f $4x^2+4x=5$

EXERCISE 1A.2

1 **a** $x = -5 \pm \sqrt{2}$ **b** no real solns. **c** $x = 4 \pm 2\sqrt{2}$

d $x = 8 \pm \sqrt{7}$ **e** $x = -3 \pm \sqrt{5}$ **f** $x = 2 \pm \sqrt{6}$

g $x = -1 \pm \sqrt{10}$ **h** $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$ **i** $x = \frac{1}{3} \pm \frac{\sqrt{7}}{3}$

2 **a** $x = 2 \pm \sqrt{3}$ **b** $x = -3 \pm \sqrt{7}$ **c** $x = 7 \pm \sqrt{3}$

d $x = 2 \pm \sqrt{7}$ **e** $x = -3 \pm \sqrt{2}$ **f** $x = 1 \pm \sqrt{7}$

g $x = -3 \pm \sqrt{11}$ **h** $x = 4 \pm \sqrt{6}$ **i** no real solns.

3 **a** $x = -1 \pm \frac{1}{\sqrt{2}}$ **b** $x = \frac{5}{2} \pm \frac{\sqrt{19}}{2}$ **c** $x = -2 \pm \sqrt{\frac{7}{3}}$

d $x = 1 \pm \sqrt{\frac{7}{3}}$ **e** $x = \frac{3}{2} \pm \sqrt{\frac{37}{20}}$ **f** $x = -\frac{1}{2} \pm \frac{\sqrt{6}}{2}$

THE QUADRATIC FORMULA

In many cases, factorising a quadratic equation or completing the square can be long or difficult. We can instead use the **quadratic formula**:

$$\text{If} \quad \boldsymbol{ax^2 + bx + c = 0}, \quad \text{then} \quad \boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

Proof:

$$\text{If} \quad ax^2 + bx + c = 0, \quad a \neq 0$$

$$\text{then} \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \{\text{dividing each term by } a, \text{ as } a \neq 0\}$$

$$\therefore \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\therefore \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \{\text{completing the square on LHS}\}$$

$$\therefore \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \{\text{factorising}\}$$

$$\therefore \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 7 **Self Tutor**

Solve for x: **a** $x^2 - 2x - 6 = 0$ **b** $2x^2 + 3x - 6 = 0$

EXERCISE 1A.3

1 Use the quadratic formula to solve exactly for x:

a $x^2 - 4x - 3 = 0$ **b** $x^2 + 6x + 7 = 0$ **c** $x^2 + 1 = 4x$

d $x^2 + 4x = 1$ **e** $x^2 - 4x + 2 = 0$ **f** $2x^2 - 2x - 3 = 0$

g $(3x + 1)^2 = -2x$ **h** $(x + 3)(2x + 1) = 9$

2 Rearrange the following equations so they are written in the form $ax^2 + bx + c = 0$, then use the quadratic formula to solve exactly for x.

a $(x + 2)(x - 1) = 2 - 3x$ **b** $(2x + 1)^2 = 3 - x$ **c** $(x - 2)^2 = 1 + x$

d $\dfrac{x - 1}{2 - x} = 2x + 1$ **e** $x - \dfrac{1}{x} = 1$ **f** $2x - \dfrac{1}{x} = 3$

EXERCISE 1A.3

1 **a** $x = 2 \pm \sqrt{7}$ **b** $x = -3 \pm \sqrt{2}$ **c** $x = 2 \pm \sqrt{3}$

d $x = -2 \pm \sqrt{5}$ **e** $x = 2 \pm \sqrt{2}$ **f** $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}$

g $x = -\frac{4}{9} \pm \frac{\sqrt{7}}{9}$ **h** $x = -\frac{7}{4} \pm \frac{\sqrt{97}}{4}$

2 **a** $x = -2 \pm 2\sqrt{2}$ **b** $x = -\frac{5}{8} \pm \frac{\sqrt{57}}{8}$ **c** $x = \frac{5}{2} \pm \frac{\sqrt{13}}{2}$

d $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}$ **e** $x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$ **f** $x = \frac{3}{4} \pm \frac{\sqrt{17}}{4}$

B THE DISCRIMINANT OF A QUADRATIC

In the quadratic formula, the quantity $b^2 - 4ac$ under the square root sign is called the **discriminant**.

The symbol **delta** Δ is used to represent the discriminant, so $\Delta = b^2 - 4ac$.

The quadratic formula becomes $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ where Δ replaces $b^2 - 4ac$.

- If $\Delta = 0$, $x = \dfrac{-b}{2a}$ is the **only solution** (a **repeated** or **double root**)
- If $\Delta > 0$, $\sqrt{\Delta}$ is a positive real number, so there are **two distinct real roots**
$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{\Delta}}{2a}$$
- If $\Delta < 0$, $\sqrt{\Delta}$ is not a real number and so there are **no real roots**.
- If a, b, and c are rational and Δ is a **square** then the equation has two rational roots which can be found by factorisation.

Example 8 **Self Tutor**

Use the discriminant to determine the nature of the roots of:

a $2x^2 - 2x + 3 = 0$ **b** $3x^2 - 4x - 2 = 0$

Example 9 — Self Tutor

Consider $x^2 - 2x + m = 0$. Find the discriminant Δ, and hence find the values of m for which the equation has:

a a repeated root **b** 2 distinct real roots **c** no real roots.

Summary:

Factorisation of quadratic	*Roots of quadratic*	*Discriminant value*
two distinct linear factors	two real distinct roots	$\Delta > 0$
two identical linear factors	two identical real roots (repeated)	$\Delta = 0$
unable to factorise	no real roots	$\Delta < 0$

Example 10 — Self Tutor

Consider the equation $kx^2 + (k+3)x = 1$. Find the discriminant Δ and draw its sign diagram. Hence, find the value of k for which the equation has:

a two distinct real roots
b two real roots
c a repeated root
d no real roots.

EXERCISE 1B

1 By using the discriminant only, state the nature of the solutions of:

a $x^2 + 7x - 3 = 0$
b $x^2 - 3x + 2 = 0$
c $3x^2 + 2x - 1 = 0$
d $5x^2 + 4x - 3 = 0$
e $x^2 + x + 5 = 0$
f $16x^2 - 8x + 1 = 0$

2 By using the discriminant only, determine which of the following quadratic equations have rational roots which can be found by factorisation.

a $6x^2 - 5x - 6 = 0$
b $2x^2 - 7x - 5 = 0$
c $3x^2 + 4x + 1 = 0$
d $6x^2 - 47x - 8 = 0$
e $4x^2 - 3x + 2 = 0$
f $8x^2 + 2x - 3 = 0$

3 For each of the following quadratic equations, determine the discriminant Δ in simplest form and draw its sign diagram. Hence find the value(s) of m for which the equation has:

i a repeated root
ii two distinct real roots
iii no real roots.

a $x^2 + 4x + m = 0$
b $mx^2 + 3x + 2 = 0$
c $mx^2 - 3x + 1 = 0$

4 For each of the following quadratic equations, find the discriminant Δ and hence draw its sign diagram. Find all values of k for which the equation has:

i two distinct real roots
ii two real roots
iii a repeated root
iv no real roots.

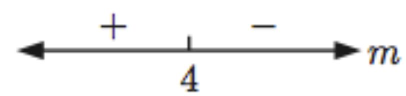
a $2x^2 + kx - k = 0$
b $kx^2 - 2x + k = 0$
c $x^2 + (k+2)x + 4 = 0$
d $2x^2 + (k-2)x + 2 = 0$
e $x^2 + (3k-1)x + (2k+10) = 0$
f $(k+1)x^2 + kx + k = 0$

EXERCISE 1B

1 **a** 2 real distinct roots **b** 2 real distinct roots
c 2 real distinct roots **d** 2 real distinct roots
e no real roots **f** a repeated root

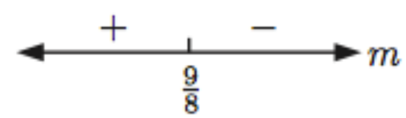
2 **a**, **c**, **d**, **f**

3 **a** $\Delta = 16 - 4m$



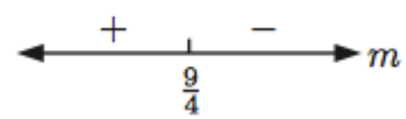
i $m = 4$ **ii** $m < 4$ **iii** $m > 4$

b $\Delta = 9 - 8m$



i $m = \frac{9}{8}$ **ii** $m < \frac{9}{8}$ **iii** $m > \frac{9}{8}$

c $\Delta = 9 - 4m$



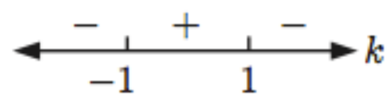
i $m = \frac{9}{4}$ **ii** $m < \frac{9}{4}$ **iii** $m > \frac{9}{4}$

4 **a** $\Delta = k^2 + 8k$

i $k < -8$ or $k > 0$ **ii** $k \leqslant -8$ or $k \geqslant 0$

iii $k = -8$ or 0 **iv** $-8 < k < 0$

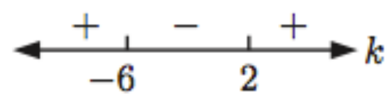
b $\Delta = 4 - 4k^2$



i $-1 < k < 1$ **ii** $-1 \leqslant k \leqslant 1$

iii $k = \pm 1$ **iv** $k < -1$ or $k > 1$

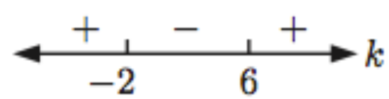
c $\Delta = k^2 + 4k - 12$



i $k < -6$ or $k > 2$ **ii** $k \leqslant -6$ or $k \geqslant 2$

iii $k = -6$ or 2 **iv** $-6 < k < 2$

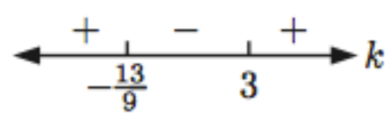
d $\Delta = k^2 - 4k - 12$



i $k < -2$ or $k > 6$ **ii** $k \leqslant -2$ or $k \geqslant 6$

iii $k = 6$ or -2 **iv** $-2 < k < 6$

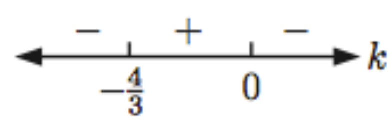
e $\Delta = 9k^2 - 14k - 39$



i $k < -\frac{13}{9}$ or $k > 3$ **ii** $k \leqslant -\frac{13}{9}$ or $k \geqslant 3$

iii $k = -\frac{13}{9}$ or 3 **iv** $-\frac{13}{9} < k < 3$

f $\Delta = -3k^2 - 4k$



i $-\frac{4}{3} < k < 0$ **ii** $-\frac{4}{3} \leqslant k \leqslant 0$

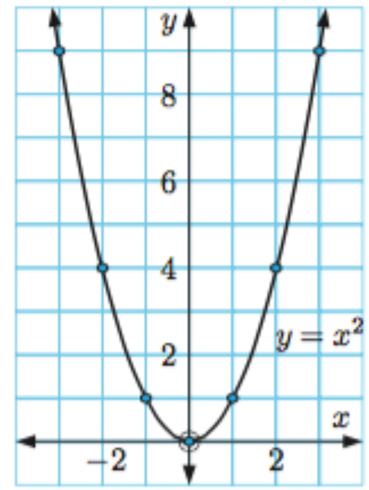
iii $k = -\frac{4}{3}$ or 0 **iv** $k < -\frac{4}{3}$ or $k > 0$

C QUADRATIC FUNCTIONS

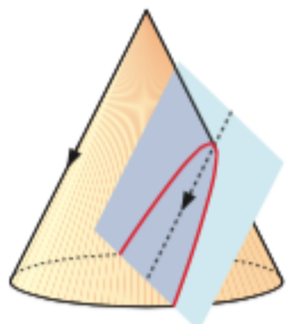
A **quadratic function** has the form $y = ax^2 + bx + c$ where $a \neq 0$.

The simplest quadratic function is $y = x^2$. Its graph can be drawn from a table of values.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9



The graph of a quadratic function is called a **parabola**.



The parabola is one of the **conic sections**, the others being circles, hyperbolae, and ellipses. They are called conic sections because they can be obtained by cutting a cone with a plane. A parabola is produced by cutting the cone with a plane parallel to its slant side.

TERMINOLOGY

The graph of a quadratic function $y = ax^2 + bx + c$, $a \neq 0$ is called a **parabola**.

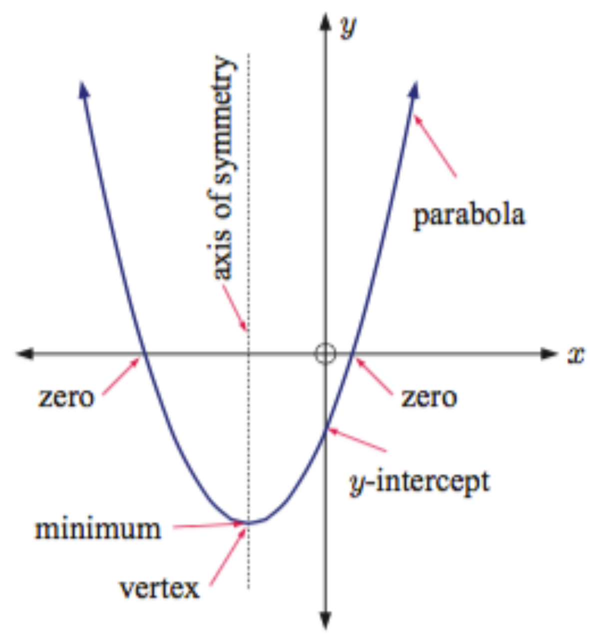
The point where the graph 'turns' is called the **vertex**.

If the graph opens upwards, the y-coordinate of the vertex is the **minimum** or **minimum turning point** and the graph is **concave upwards**.

If the graph opens downwards, the y-coordinate of the vertex is the **maximum** or **maximum turning point** and the graph is **concave downwards**.

The vertical line that passes through the vertex is called the **axis of symmetry**. Every parabola is symmetrical about its axis of symmetry.

The point where the graph crosses the y-axis is the **y-intercept**.



The points (if they exist) where the graph crosses the x-axis are called the **x-intercepts**. They correspond to the **roots** of the equation $y = 0$.

INVESTIGATION 1 GRAPHING $y = a(x-p)(x-q)$

This investigation is best done using a **graphing package** or **graphics calculator**.

GRAPHING PACKAGE

What to do:

1 **a** Use technology to help you to sketch:
$y=(x-1)(x-3)$, $y=2(x-1)(x-3)$, $y=-(x-1)(x-3)$,
$y=-3(x-1)(x-3)$ and $y=-\frac{1}{2}(x-1)(x-3)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of a in $y=a(x-1)(x-3)$?

2 **a** Use technology to help you to sketch:
$y=2(x-1)(x-4)$, $y=2(x-3)(x-5)$, $y=2(x+1)(x-2)$,
$y=2x(x+5)$ and $y=2(x+2)(x+4)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of p and q in $y=2(x-p)(x-q)$?

3 **a** Use technology to help you to sketch:
$y=2(x-1)^2$, $y=2(x-3)^2$, $y=2(x+2)^2$, $y=2x^2$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of p in $y=2(x-p)^2$?

4 Copy and complete:

- If a quadratic has the form $y=a(x-p)(x-q)$ then it ...... the x-axis at ......
- If a quadratic has the form $y=a(x-p)^2$ then it ...... the x-axis at ......

INVESTIGATION 2 GRAPHING $y = a(x-h)^2 + k$

This investigation is also best done using technology.

GRAPHING PACKAGE

What to do:

1 **a** Use technology to help you to sketch:
$y = (x-3)^2 + 2$, $y = 2(x-3)^2 + 2$, $y = -2(x-3)^2 + 2$,
$y = -(x-3)^2 + 2$ and $y = -\frac{1}{3}(x-3)^2 + 2$

b Find the coordinates of the vertex for each function in **a**.

c What is the geometrical significance of a in $y = a(x-3)^2 + 2$?

2 **a** Use technology to help you to sketch:
$y = 2(x-1)^2 + 3$, $y = 2(x-2)^2 + 4$, $y = 2(x-3)^2 + 1$,
$y = 2(x+1)^2 + 4$, $y = 2(x+2)^2 - 5$ and $y = 2(x+3)^2 - 2$

b Find the coordinates of the vertex for each function in **a**.

c What is the geometrical significance of h and k in $y = 2(x-h)^2 + k$?

3 Copy and complete:
If a quadratic has the form $y = a(x-h)^2 + k$ then its vertex has coordinates ......
The graph of $y = a(x-h)^2 + k$ is a ...... of the graph of $y = ax^2$ with vector ......

From **Investigations 1** and **2** you should have discovered that a, the coefficient of x^2, controls the width of the graph and whether it opens upwards or downwards.

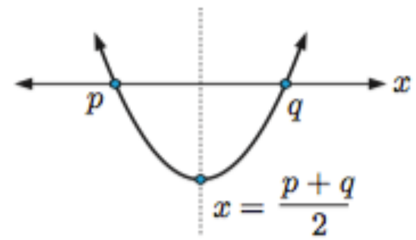
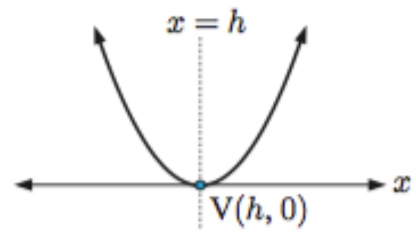
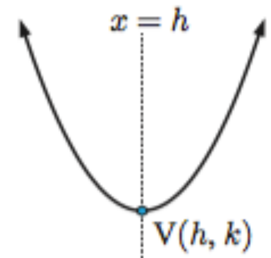
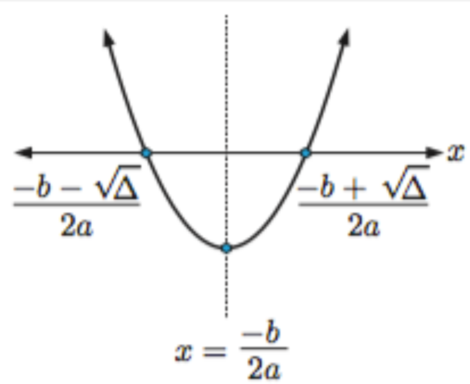
For a quadratic function $y = ax^2 + bx + c$, $a \neq 0$:

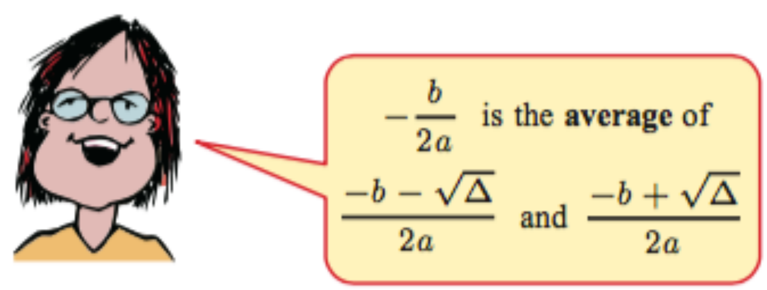
- $a > 0$ produces the shape  called concave up.

  $a < 0$ produces the shape  called concave down.

- If $-1 < a < 1$, $a \neq 0$ the graph is wider than $y = x^2$.
  If $a < -1$ or $a > 1$ the graph is narrower than $y = x^2$.

Summary:

Quadratic form, $a \neq 0$	*Graph*	*Facts*
• $y = a(x-p)(x-q)$ p, q are real		x-intercepts are p and q axis of symmetry is $x = \frac{p+q}{2}$ vertex is $\left(\frac{p+q}{2}, f(\frac{p+q}{2})\right)$
• $y = a(x-h)^2$ h is real		touches x-axis at h axis of symmetry is $x = h$ vertex is $(h, 0)$
• $y = a(x-h)^2 + k$		axis of symmetry is $x = h$ vertex is (h, k)
• $y = ax^2 + bx + c$		y-intercept c axis of symmetry is $x = \frac{-b}{2a}$ vertex is $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$ x-intercepts for $\Delta \geqslant 0$ are $\frac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$

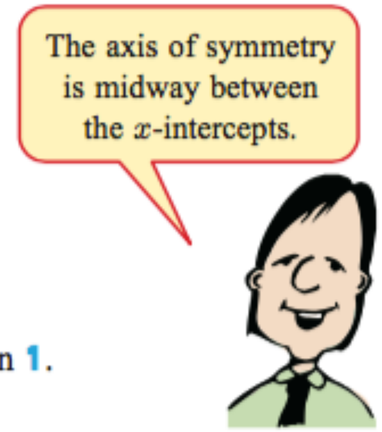


Example 11 **Self Tutor**

Using axes intercepts only, sketch the graphs of:

a $y = 2(x+3)(x-1)$ **b** $y = -2(x-1)(x-2)$ **c** $y = \frac{1}{2}(x+2)^2$

EXERCISE 1C.1



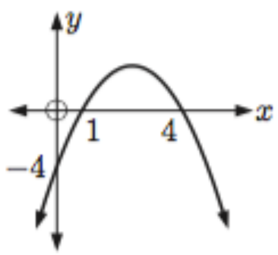
1 Using axes intercepts only, sketch the graphs of:

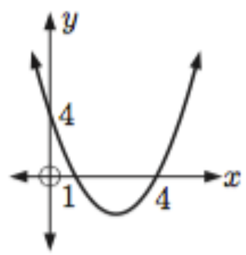
- **a** $y = (x-4)(x+2)$
- **b** $y = -(x-4)(x+2)$
- **c** $y = 2(x+3)(x+5)$
- **d** $y = -3x(x+4)$
- **e** $y = 2(x+3)^2$
- **f** $y = -\frac{1}{4}(x+2)^2$

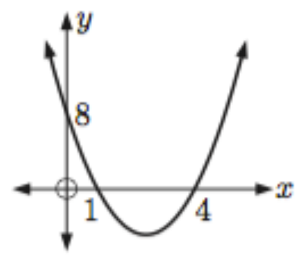
2 State the equation of the axis of symmetry for each graph in question **1**.

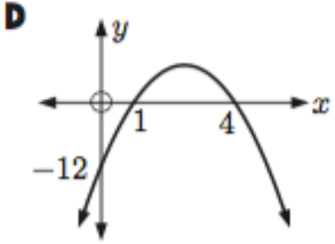
3 Match each quadratic function with its corresponding graph.

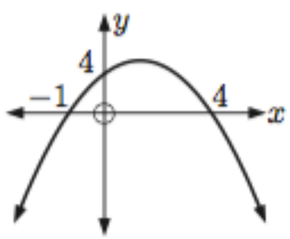
- **a** $y = 2(x-1)(x-4)$
- **b** $y = -(x+1)(x-4)$
- **c** $y = (x-1)(x-4)$
- **d** $y = (x+1)(x-4)$
- **e** $y = 2(x+4)(x-1)$
- **f** $y = -3(x+4)(x-1)$
- **g** $y = -(x-1)(x-4)$
- **h** $y = -3(x-1)(x-4)$

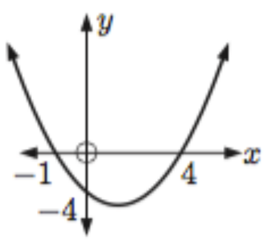
A


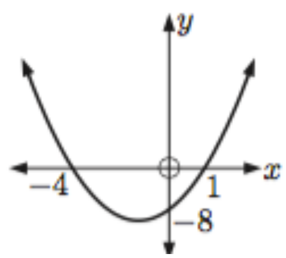
B


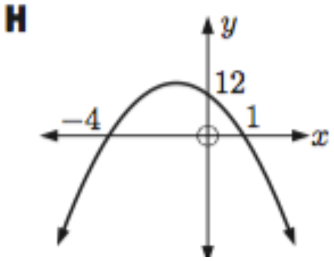
C


D


E


F


G


H


Example 12 **Self Tutor**

Use the vertex, axis of symmetry, and y-intercept to graph $y = -2(x+1)^2 + 4$.

4 Use the vertex, axis of symmetry, and y-intercept to graph:

a $y=(x-1)^2+3$ **b** $y=2(x+2)^2+1$ **c** $y=-2(x-1)^2-3$

d $y=\frac{1}{2}(x-3)^2+2$ **e** $y=-\frac{1}{3}(x-1)^2+4$ **f** $y=-\frac{1}{10}(x+2)^2-3$

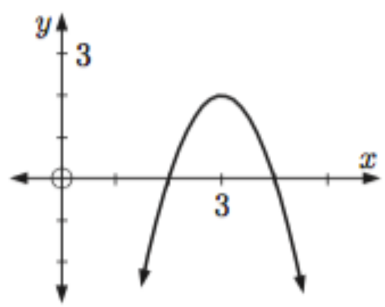
5 Match each quadratic function with its corresponding graph:

a $y=-(x+1)^2+3$ **b** $y=-2(x-3)^2+2$ **c** $y=x^2+2$

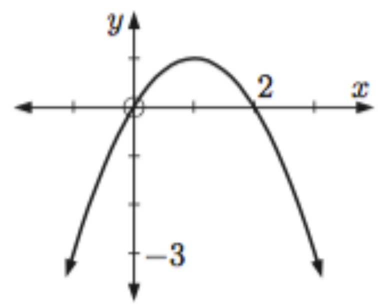
d $y=-(x-1)^2+1$ **e** $y=(x-2)^2-2$ **f** $y=\frac{1}{3}(x+3)^2-3$

g $y=-x^2$ **h** $y=-\frac{1}{2}(x-1)^2+1$ **i** $y=2(x+2)^2-1$

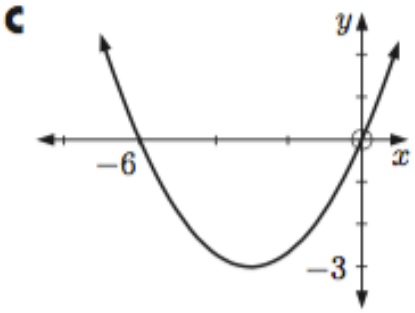
A



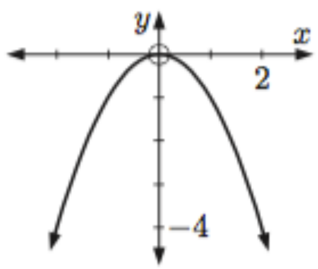
B



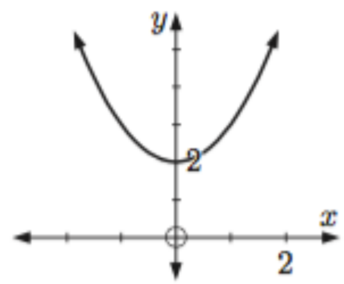
C



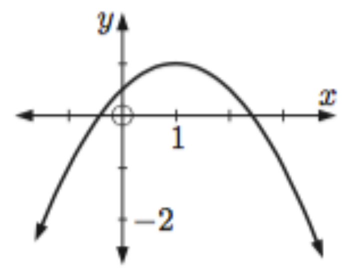
D



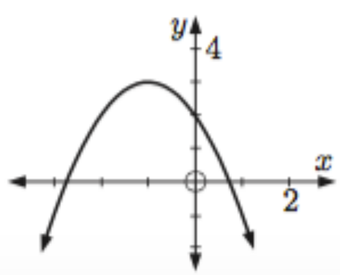
E



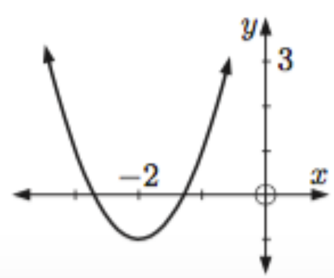
F



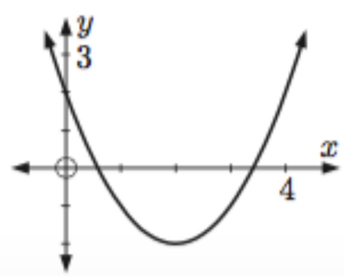
G



H



I



Example 13 **Self Tutor**

Determine the coordinates of the vertex of $y = 2x^2 - 8x + 1$.

6 Locate the turning point or vertex for each of the following quadratic functions:

a $y = x^2 - 4x + 2$ **b** $y = x^2 + 2x - 3$ **c** $y = 2x^2 + 4$

d $y = -3x^2 + 1$ **e** $y = 2x^2 + 8x - 7$ **f** $y = -x^2 - 4x - 9$

g $y = 2x^2 + 6x - 1$ **h** $y = 2x^2 - 10x + 3$ **i** $y = -\frac{1}{2}x^2 + x - 5$

7 Find the x-intercepts for:

a $y = x^2 - 9$ **b** $y = 2x^2 - 6$ **c** $y = x^2 + 7x + 10$

d $y = x^2 + x - 12$ **e** $y = 4x - x^2$ **f** $y = -x^2 - 6x - 8$

g $y = -2x^2 - 4x - 2$ **h** $y = 4x^2 - 24x + 36$ **i** $y = x^2 - 4x + 1$

j $y = x^2 + 4x - 3$ **k** $y = x^2 - 6x - 2$ **l** $y = x^2 + 8x + 11$

Example 14 **Self Tutor**

Consider the quadratic $y = 2x^2 + 6x - 3$.

a State the axis of symmetry.

b Find the coordinates of the vertex.

c Find the axes intercepts.

d Hence, sketch the quadratic.

8 For each of the following quadratics:

i state the axis of symmetry

ii find the coordinates of the vertex

iii find the axes intercepts, if they exist

iv sketch the quadratic.

a $y = x^2 - 2x + 5$

b $y = x^2 + 4x - 1$

c $y = 2x^2 - 5x + 2$

d $y = -x^2 + 3x - 2$

e $y = -3x^2 + 4x - 1$

f $y = -2x^2 + x + 1$

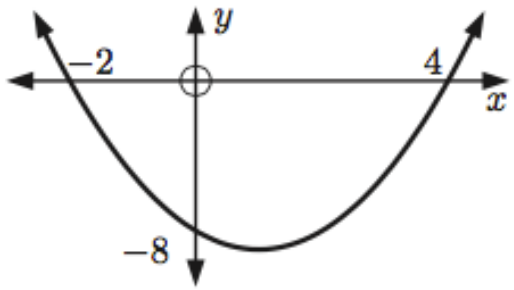
g $y = 6x - x^2$

h $y = -x^2 - 6x - 8$

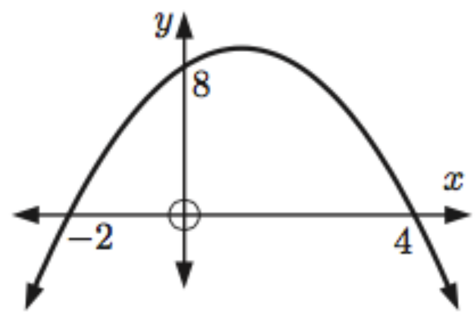
i $y = -\frac{1}{4}x^2 + 2x + 1$

EXERCISE 1C.1

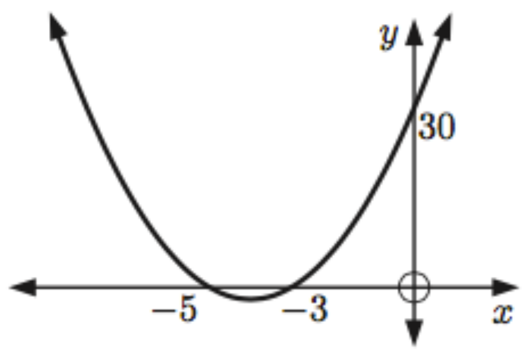
1 a $y = (x-4)(x+2)$



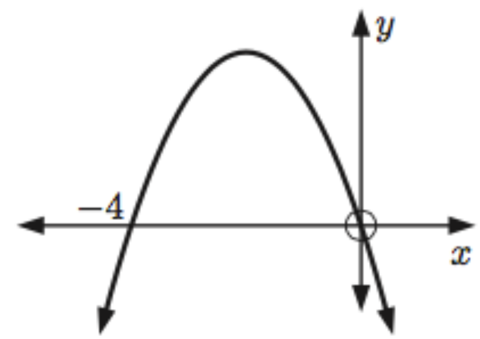
b $y = -(x-4)(x+2)$



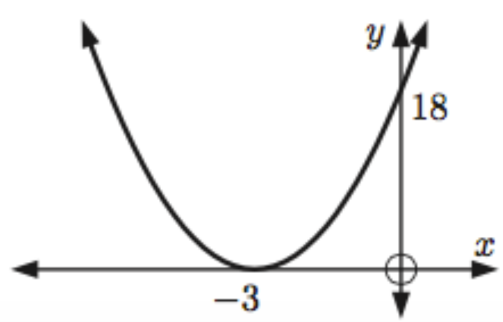
c $y = 2(x+3)(x+5)$



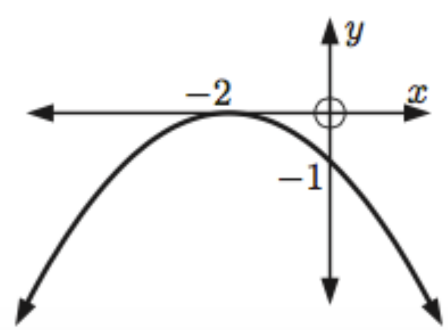
d $y = -3x(x+4)$



e $y = 2(x+3)^2$



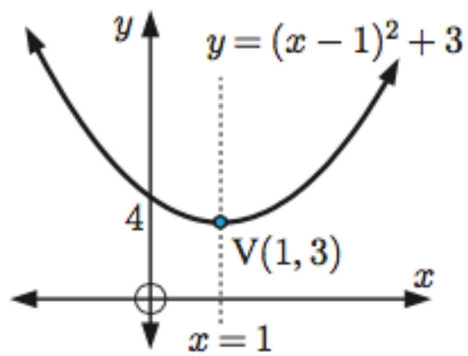
f $y = -\frac{1}{4}(x+2)^2$



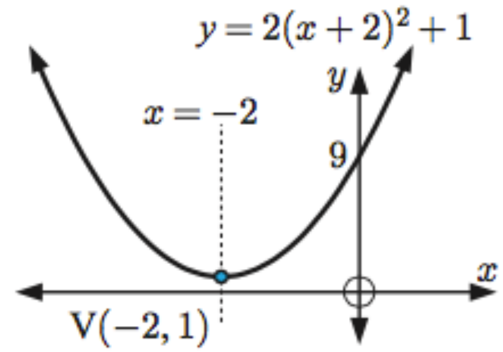
2 **a** $x = 1$ **b** $x = 1$ **c** $x = -4$
d $x = -2$ **e** $x = -3$ **f** $x = -2$

3 **a** C **b** E **c** B **d** F **e** G **f** H **g** A **h** D

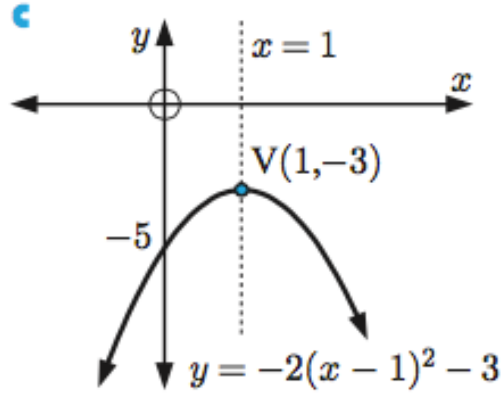
4 **a**



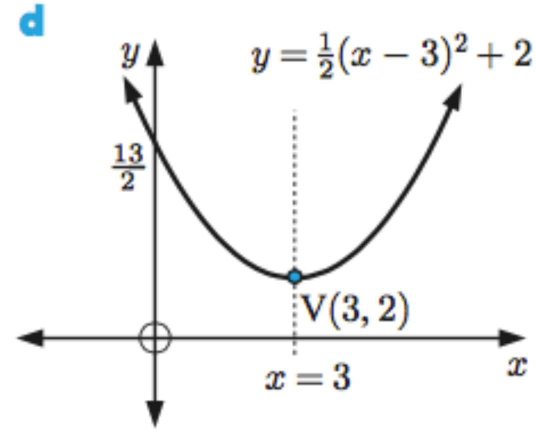
b



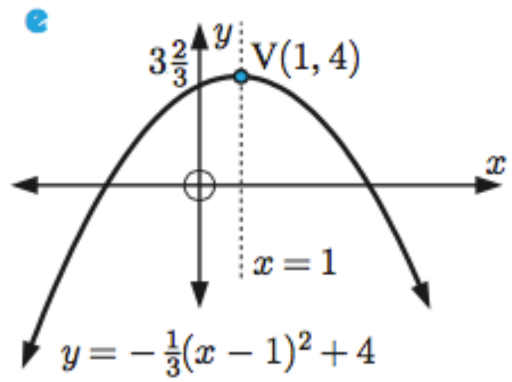
c



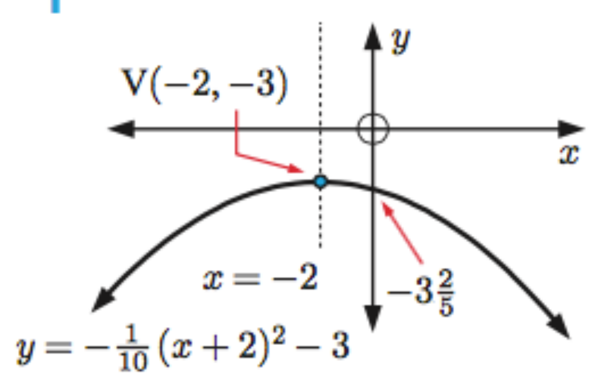
d



e



f



5 **a** **G** **b** **A** **c** **E** **d** **B** **e** **I**
f **C** **g** **D** **h** **F** **i** **H**

6 **a** $(2, -2)$ **b** $(-1, -4)$ **c** $(0, 4)$
d $(0, 1)$ **e** $(-2, -15)$ **f** $(-2, -5)$
g $(-\frac{3}{2}, -\frac{11}{2})$ **h** $(\frac{5}{2}, -\frac{19}{2})$ **i** $(1, -\frac{9}{2})$

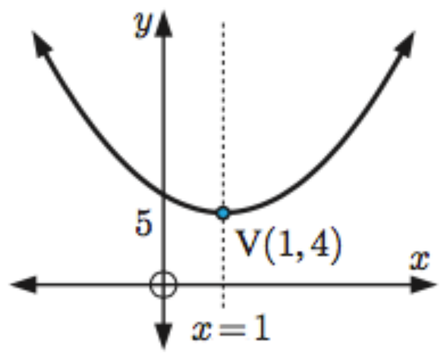
7 **a** ± 3 **b** $\pm\sqrt{3}$ **c** -5 and -2
d 3 and -4 **e** 0 and 4 **f** -4 and -2
g -1 (touching) **h** 3 (touching) **i** $2 \pm \sqrt{3}$
j $-2 \pm \sqrt{7}$ **k** $3 \pm \sqrt{11}$ **l** $-4 \pm \sqrt{5}$

8 **a** **i** $x = 1$

ii $(1, 4)$

iii no x-intercept, y-intercept 5

iv

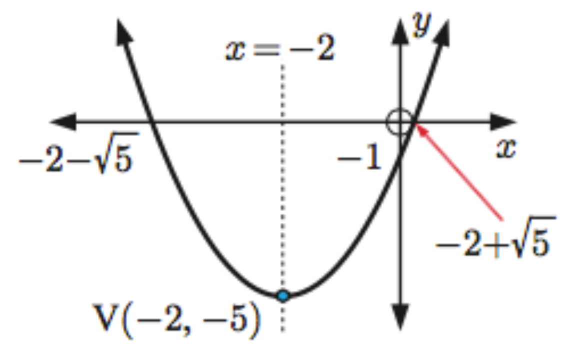


b **i** $x = -2$

ii $(-2, -5)$

iii x-int. $-2 \pm \sqrt{5}$, y-intercept -1

iv

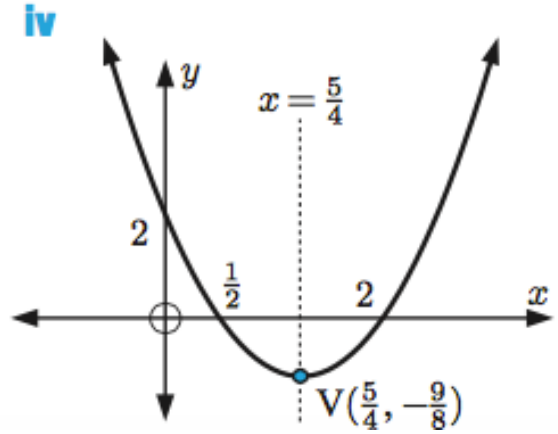


c **i** $x = \frac{5}{4}$

ii $(\frac{5}{4}, -\frac{9}{8})$

iii x-intercepts $\frac{1}{2}$, 2, y-intercept 2

iv

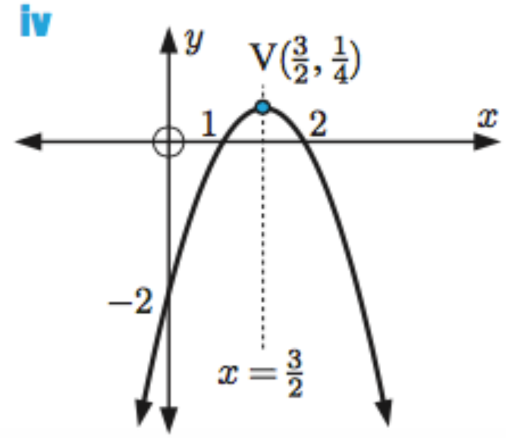


d **i** $x = \frac{3}{2}$

ii $(\frac{3}{2}, \frac{1}{4})$

iii x-intercepts 1, 2, y-intercept -2

iv

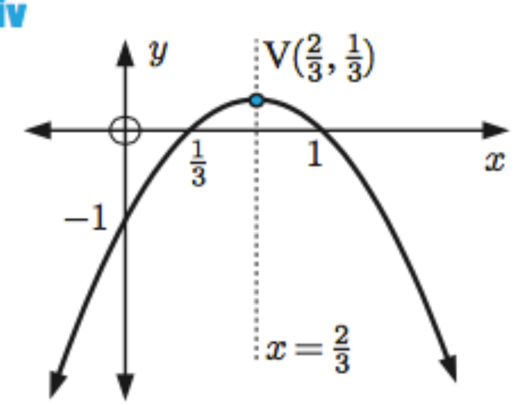


e **i** $x = \frac{2}{3}$

ii $(\frac{2}{3}, \frac{1}{3})$

iii x-intercepts $\frac{1}{3}$, 1,
y-intercept -1

iv

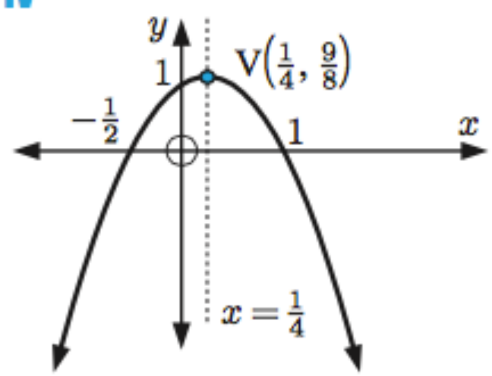


f **i** $x = \frac{1}{4}$

ii $(\frac{1}{4}, \frac{9}{8})$

iii x-intercepts $-\frac{1}{2}$, 1,
y-intercept 1

iv

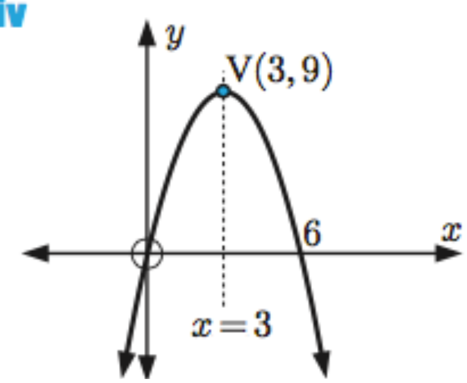


g **i** $x = 3$

ii (3, 9)

iii x-intercepts 0, 6,
y-intercept 0

iv

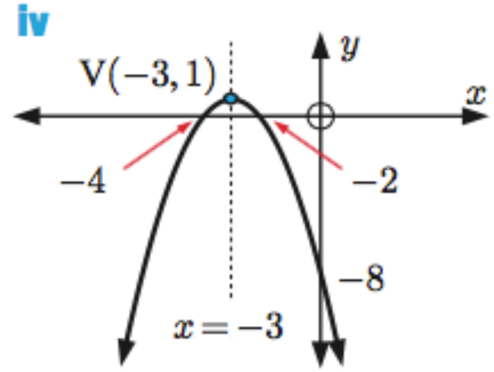


h **i** $x = -3$

ii $(-3, 1)$

iii x-int. -2, -4,
y-intercept -8

iv

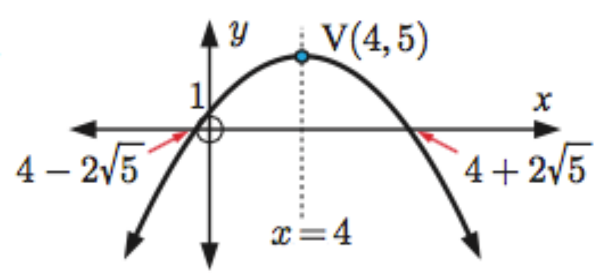


i **i** $x = 4$

ii (4, 5)

iii x-int. $4 \pm 2\sqrt{5}$,
y-intercept 1

iv



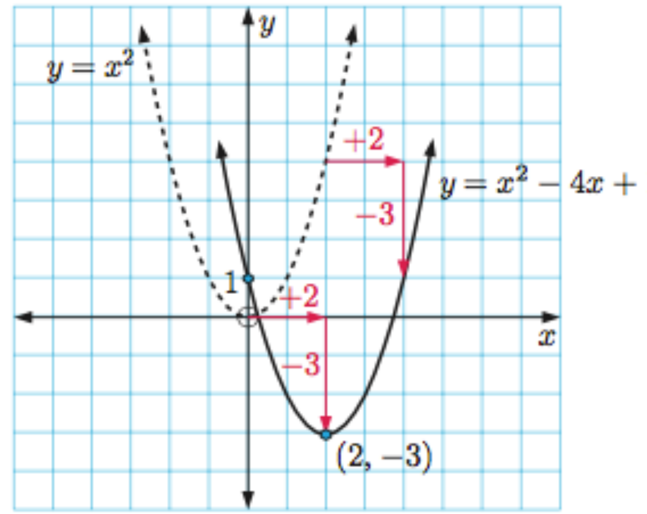
SKETCHING GRAPHS BY 'COMPLETING THE SQUARE'

If we wish to find the vertex of a quadratic given in general form $y = ax^2 + bx + c$ then one approach is to convert it to the form $y = a(x-h)^2 + k$ where we can read off the coordinates of the vertex (h, k). One way to do this is to **'complete the square'**.

Consider the simple case $y = x^2 - 4x + 1$, for which $a = 1$.

$$y = x^2 - 4x + 1$$
$$\therefore\ y = \underbrace{x^2 - 4x + 2^2}\ \underbrace{+\,1 - 2^2}$$
$$\therefore\ y = (x-2)^2 \quad - 3$$

To obtain the graph of $y = x^2 - 4x + 1$ from the graph of $y = x^2$, we shift it 2 units to the right and 3 units down.



Example 15 **Self Tutor**

Write $y = x^2 + 4x + 3$ in the form $y = (x - h)^2 + k$ by 'completing the square'. Hence sketch $y = x^2 + 4x + 3$, stating the coordinates of the vertex.

EXERCISE 1C.2

1 Write the following quadratics in the form $y = (x - h)^2 + k$ by 'completing the square'. Hence sketch each function, stating the coordinates of the vertex.

a $y = x^2 - 2x + 3$

b $y = x^2 + 4x - 2$

c $y = x^2 - 4x$

d $y = x^2 + 3x$

e $y = x^2 + 5x - 2$

f $y = x^2 - 3x + 2$

g $y = x^2 - 6x + 5$

h $y = x^2 + 8x - 2$

i $y = x^2 - 5x + 1$

Example 16 **Self Tutor**

a Convert $y = 3x^2 - 4x + 1$ to the form $y = a(x - h)^2 + k$ without technology.

b Hence, write down the coordinates of its vertex and sketch the quadratic. Use technology to check your answer.

2 For each of the following quadratics:

- **i** Write the quadratic in the form $y = a(x-h)^2 + k$ without using technology.
- **ii** State the coordinates of the vertex.
- **iii** Find the y-intercept.
- **iv** Sketch the graph of the quadratic.
- **v** Use technology to check your answers.

a $y = 2x^2 + 4x + 5$ **b** $y = 2x^2 - 8x + 3$

c $y = 2x^2 - 6x + 1$ **d** $y = 3x^2 - 6x + 5$

e $y = -x^2 + 4x + 2$ **f** $y = -2x^2 - 5x + 3$

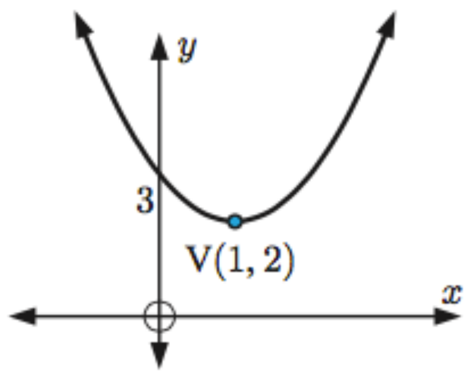
3 Use the **graphing package** or your **graphics calculator** to determine the vertex of each of the following functions. *Hence* write each function in the form $y = a(x-h)^2 + k$.

a $y = x^2 - 4x + 7$ **b** $y = x^2 + 6x + 3$ **c** $y = -x^2 + 4x + 5$

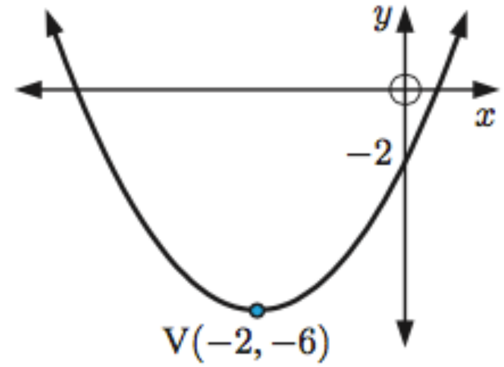
d $y = 2x^2 + 6x - 4$ **e** $y = -2x^2 - 10x + 1$ **f** $y = 3x^2 - 9x - 5$

EXERCISE 1C.2

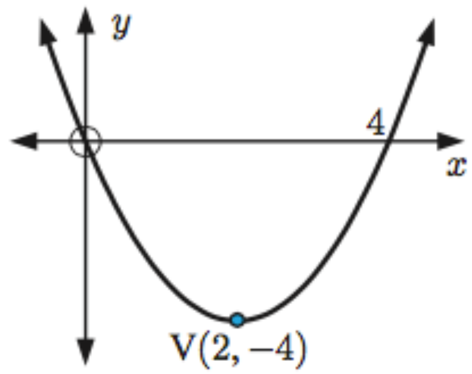
1 **a** $y = (x-1)^2 + 2$



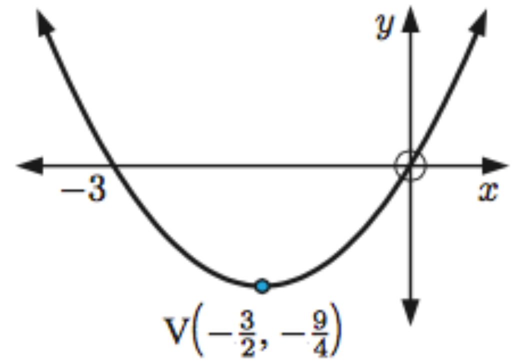
b $y = (x+2)^2 - 6$



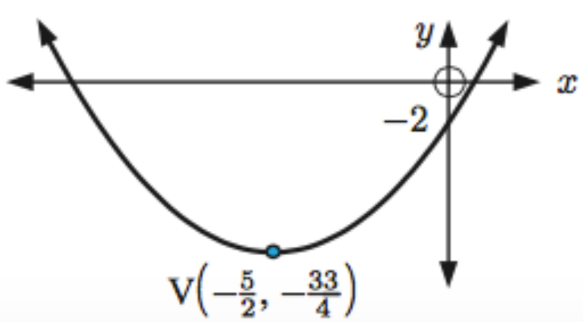
c $y = (x-2)^2 - 4$



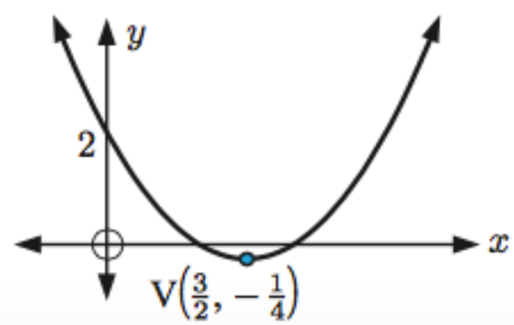
d $y = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$



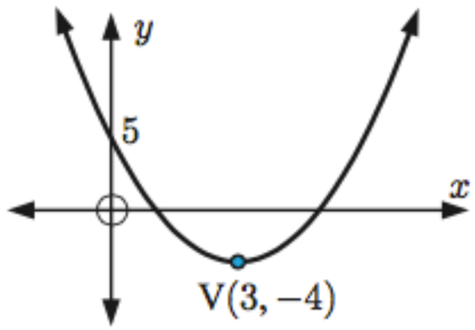
e $y = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$



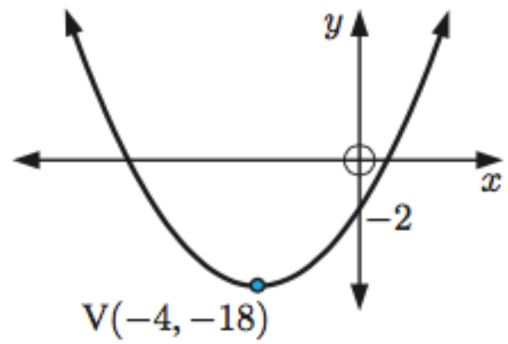
f $y = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$



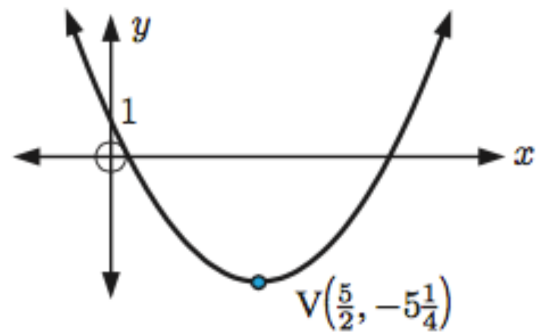
g $y = (x-3)^2 - 4$



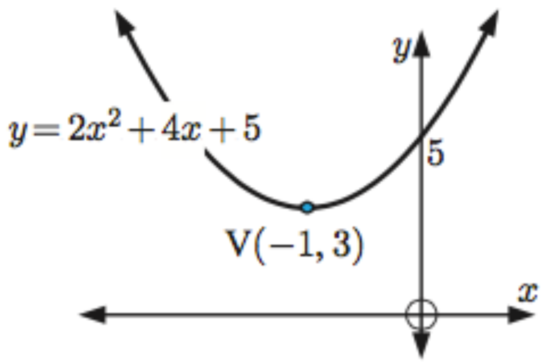
h $y = (x+4)^2 - 18$



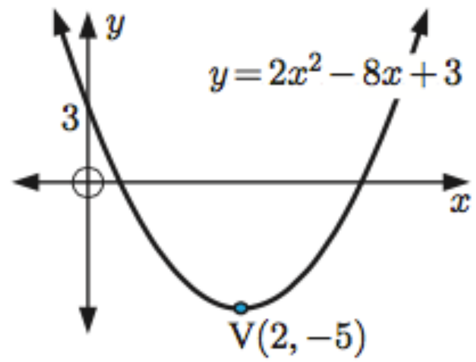
i $y = \left(x - \frac{5}{2}\right)^2 - 5\frac{1}{4}$



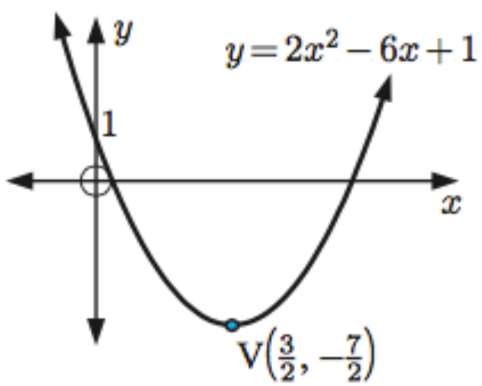
2 **a** **i** $y = 2(x+1)^2 + 3$
ii $(-1, 3)$ **iii** 5
iv



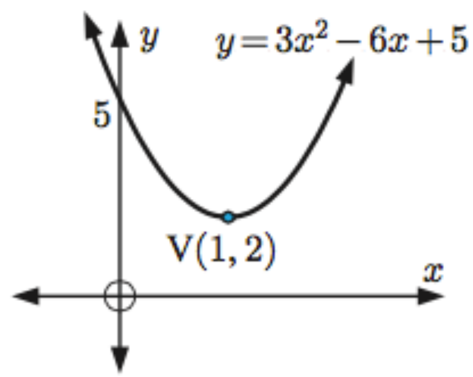
b **i** $y = 2(x-2)^2 - 5$
ii $(2, -5)$ **iii** 3
iv



c **i** $y = 2(x - \frac{3}{2})^2 - \frac{7}{2}$
ii $(\frac{3}{2}, -\frac{7}{2})$ **iii** 1
iv



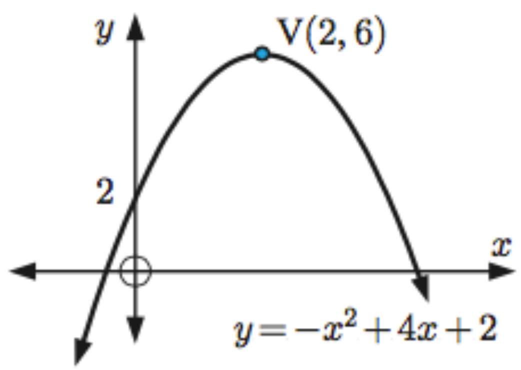
d **i** $y = 3(x-1)^2 + 2$
ii $(1, 2)$ **iii** 5
iv



e **i** $y = -(x-2)^2 + 6$

ii $(2, 6)$ **iii** 2

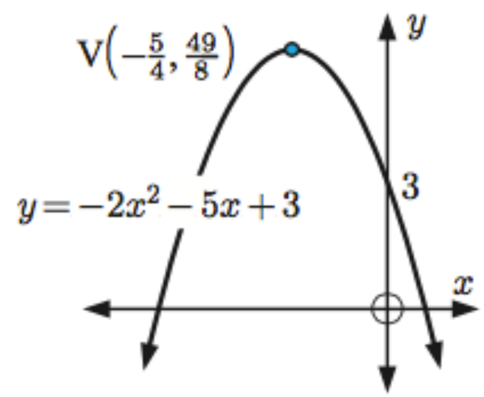
iv



f **i** $y = -2(x + \frac{5}{4})^2 + \frac{49}{8}$

ii $(-\frac{5}{4}, \frac{49}{8})$ **iii** 3

iv



3 **a** $y = (x-2)^2 + 3$

b $y = (x+3)^2 - 6$

c $y = -(x-2)^2 + 9$

d $y = 2\left(x + \frac{3}{2}\right)^2 - \frac{17}{2}$

e $y = -2\left(x + \frac{5}{2}\right)^2 + \frac{27}{2}$

f $y = 3\left(x - \frac{3}{2}\right)^2 - \frac{47}{4}$

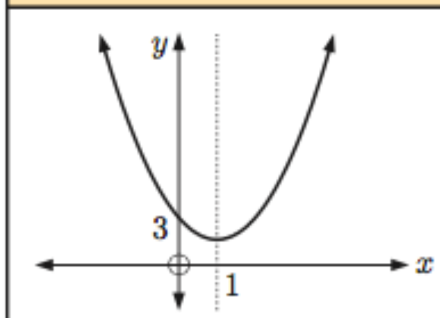
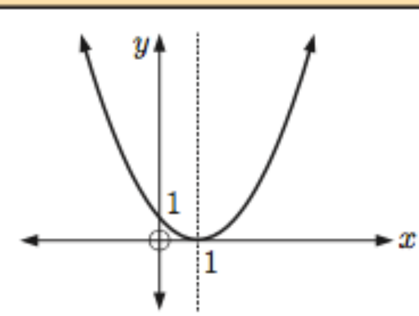
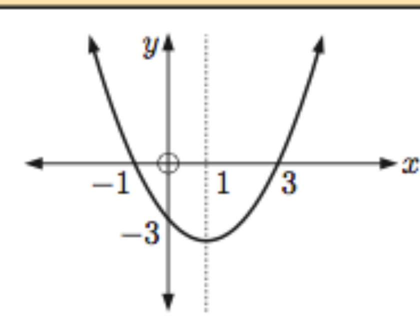
THE DISCRIMINANT AND THE QUADRATIC GRAPH

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $\Delta = b^2 - 4ac$.

We used Δ to determine the number of real roots of the equation. If they exist, these roots correspond to zeros of the quadratic $y = ax^2 + bx + c$. Δ therefore tells us about the relationship between a quadratic function and the x-axis.

The graphs of $y = x^2 - 2x + 3$, $y = x^2 - 2x + 1$, and $y = x^2 - 2x - 3$ all have the same axis of symmetry, $x = 1$.

Consider the following table:

<table>
<tr><th>$y = x^2 - 2x + 3$</th><th>$y = x^2 - 2x + 1$</th><th>$y = x^2 - 2x - 3$</th></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>$\Delta = b^2 - 4ac$
$= (-2)^2 - 4(1)(3)$
$= -8$</td><td>$\Delta = b^2 - 4ac$
$= (-2)^2 - 4(1)(1)$
$= 0$</td><td>$\Delta = b^2 - 4ac$
$= (-2)^2 - 4(1)(-3)$
$= 16$</td></tr>
<tr><td>$\Delta < 0$</td><td>$\Delta = 0$</td><td>$\Delta > 0$</td></tr>
<tr><td>does not cut the x-axis</td><td>touches the x-axis</td><td>cuts the x-axis twice</td></tr>
</table>

For a quadratic function $y = ax^2 + bx + c$, we consider the discriminant $\Delta = b^2 - 4ac$.

If $\Delta < 0$, the graph does not cut the x-axis.
If $\Delta = 0$, the graph *touches* the x-axis.
If $\Delta > 0$, the graph cuts the x-axis twice.

POSITIVE DEFINITE AND NEGATIVE DEFINITE QUADRATICS

Positive definite quadratics are quadratics which are positive for all values of x. So, $ax^2 + bx + c > 0$ for all $x \in \mathbb{R}$.

Test: A quadratic is **positive definite** if and only if $a > 0$ and $\Delta < 0$.

Negative definite quadratics are quadratics which are negative for all values of x. So, $ax^2 + bx + c < 0$ for all $x \in \mathbb{R}$.

Test: A quadratic is **negative definite** if and only if $a < 0$ and $\Delta < 0$.

Example 17 **Self Tutor**

Use the discriminant to determine the relationship between the graph of each function and the x-axis:

a $y = x^2 + 3x + 4$

b $y = -2x^2 + 5x + 1$

EXERCISE 1C.3

1 Use the discriminant to determine the relationship between the graph and x-axis for:

a $y = x^2 + x - 2$
b $y = x^2 - 4x + 1$
c $y = -x^2 - 3$
d $y = x^2 + 7x - 2$
e $y = x^2 + 8x + 16$
f $y = -2x^2 + 3x + 1$
g $y = 6x^2 + 5x - 4$
h $y = -x^2 + x + 6$
i $y = 9x^2 + 6x + 1$

2 Show that:

a $x^2 - 3x + 6 > 0$ for all x
b $4x - x^2 - 6 < 0$ for all x
c $2x^2 - 4x + 7$ is positive definite
d $-2x^2 + 3x - 4$ is negative definite.

3 Explain why $3x^2 + kx - 1$ is never positive definite for any value of k.

4 Under what conditions is $2x^2 + kx + 2$ positive definite?

EXERCISE 1C.3

1 **a** cuts x-axis twice, concave up

b cuts x-axis twice, concave up

c lies entirely below the x-axis, concave down, negative definite

d cuts x-axis twice, concave up

e touches x-axis, concave up

f cuts x-axis twice, concave down

g cuts x-axis twice, concave up

h cuts x-axis twice, concave down

i touches x-axis, concave up

2 **a** $a = 1$ which is > 0 and $\Delta = -15$ which is < 0
so is entirely above the x-axis.

b $a = -1$ which is < 0 and $\Delta = -8$ which is < 0
so is entirely below the x-axis.

c $a = 2$ which is > 0 and $\Delta = -40$ which is < 0
so is entirely above the x-axis.

d $a = -2$ which is < 0 and $\Delta = -23$ which is < 0
so is entirely below the x-axis.

3 $a = 3$ which is > 0 and $\Delta = k^2 + 12$ which is always > 0
{as $k^2 \geqslant 0$ for all k} $\therefore$ always cuts x-axis twice.

4 $-4 < k < 4$

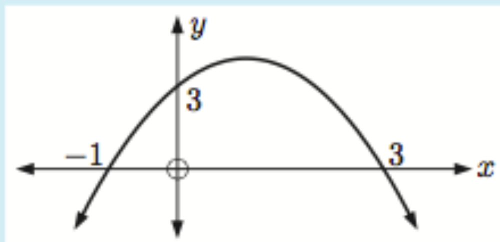
D FINDING A QUADRATIC FROM ITS GRAPH

If we are given sufficient information on or about a graph we can determine the quadratic function in whatever form is required.

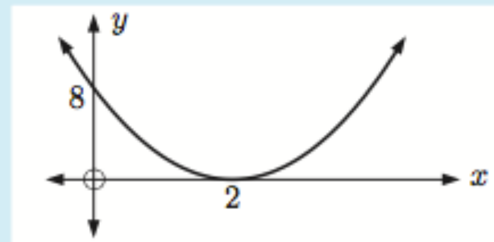
Example 18 — Self Tutor

Find the equation of the quadratic function with graph:

a

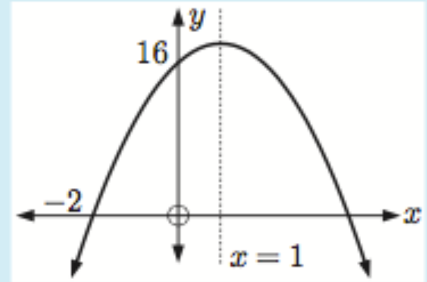


b



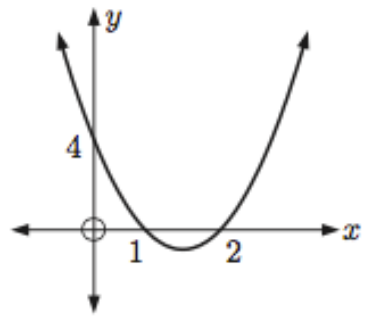
Example 19 **Self Tutor**

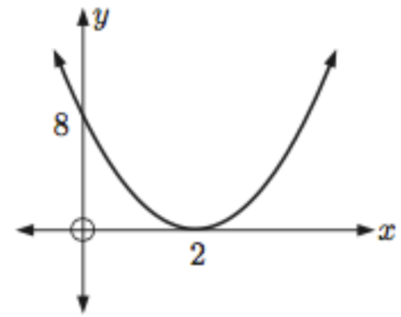
Find the equation of the quadratic function with graph:

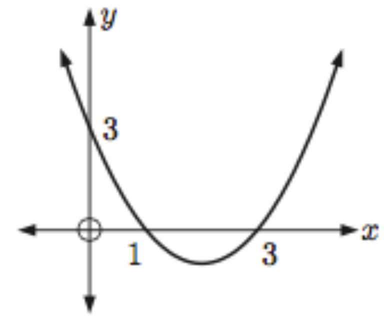


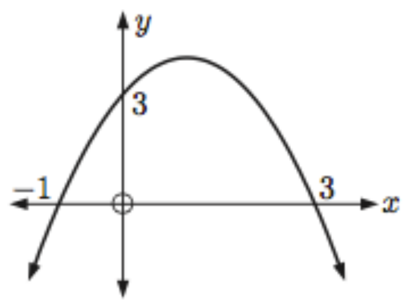
EXERCISE 1D

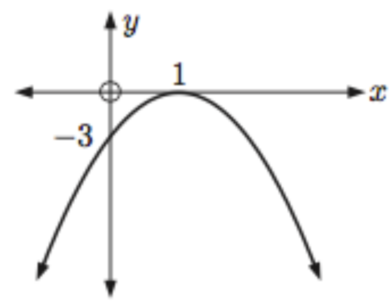
1 Find the equation of the quadratic with graph:

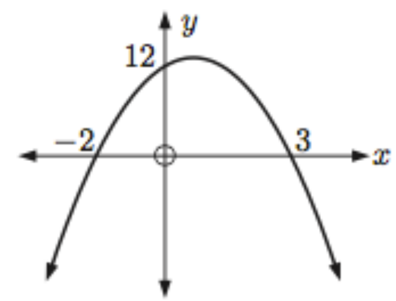
a


b


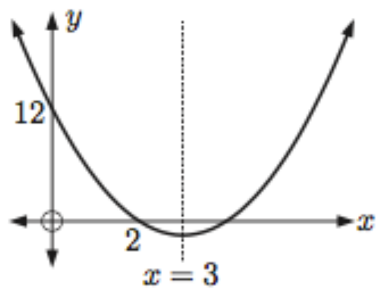
c


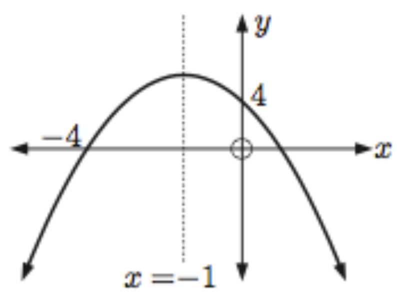
d


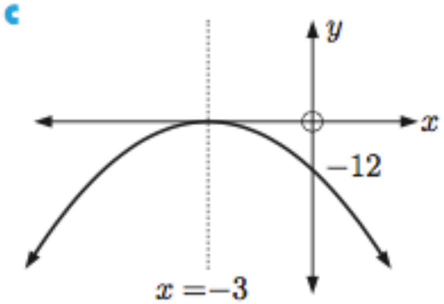
e


f


2 Find the quadratic with graph:

a


b


c


Example 20 **Self Tutor**

Find the equation of the quadratic whose graph cuts the x-axis at 4 and -3, and which passes through the point $(2, -20)$. Give your answer in the form $y = ax^2 + bx + c$.

3 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph:

a cuts the x-axis at 5 and 1, and passes through $(2, -9)$

b cuts the x-axis at 2 and $-\frac{1}{2}$, and passes through $(3, -14)$

c touches the x-axis at 3 and passes through $(-2, -25)$

d touches the x-axis at -2 and passes through $(-1, 4)$

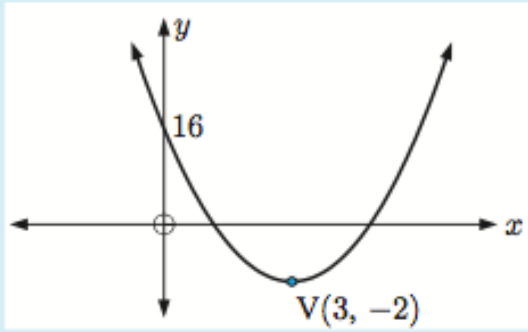
e cuts the x-axis at 3, passes through $(5, 12)$ and has axis of symmetry $x = 2$

f cuts the x-axis at 5, passes through $(2, 5)$ and has axis of symmetry $x = 1$.

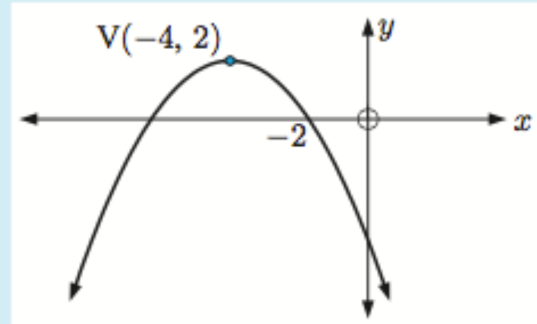
Example 21 **Self Tutor**

Find the equation of each quadratic function given its graph:

a

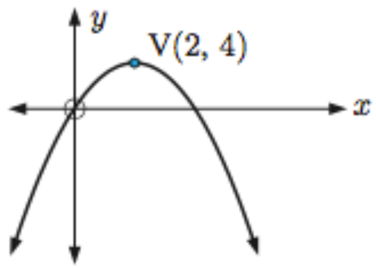


b

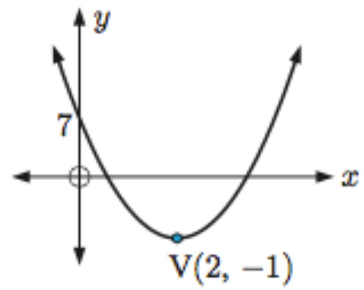


4 If V is the vertex, find the equation of the quadratic function with graph:

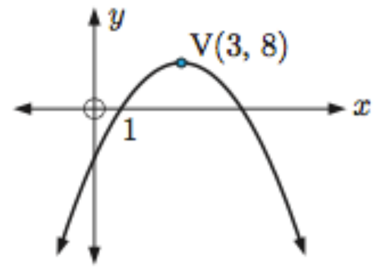
a



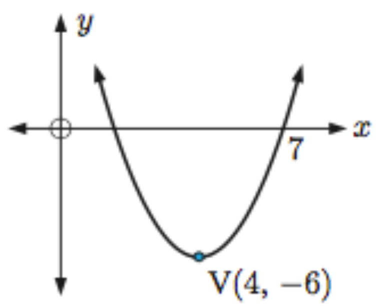
b



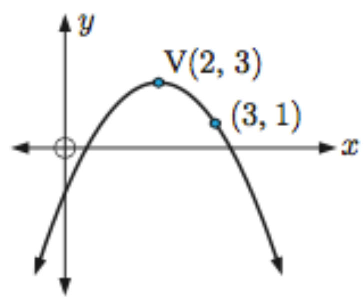
c



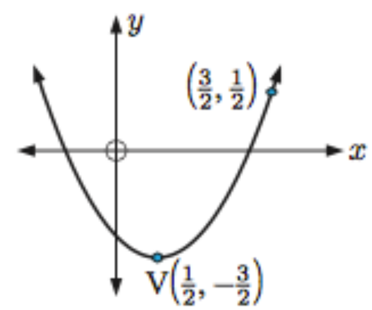
d



e



f



EXERCISE 1D

1 a $y = 2(x-1)(x-2)$
b $y = 2(x-2)^2$
c $y = (x-1)(x-3)$
d $y = -(x-3)(x+1)$
e $y = -3(x-1)^2$
f $y = -2(x+2)(x-3)$

2 a $y = \frac{3}{2}(x-2)(x-4)$
b $y = -\frac{1}{2}(x+4)(x-2)$
c $y = -\frac{4}{3}(x+3)^2$

3 a $y = 3x^2 - 18x + 15$
b $y = -4x^2 + 6x + 4$
c $y = -x^2 + 6x - 9$
d $y = 4x^2 + 16x + 16$
e $y = \frac{3}{2}x^2 - 6x + \frac{9}{2}$
f $y = -\frac{1}{3}x^2 + \frac{2}{3}x + 5$

4 a $y = -(x-2)^2 + 4$
b $y = 2(x-2)^2 - 1$
c $y = -2(x-3)^2 + 8$
d $y = \frac{2}{3}(x-4)^2 - 6$
e $y = -2(x-2)^2 + 3$
f $y = 2(x-\frac{1}{2})^2 - \frac{3}{2}$

E WHERE FUNCTIONS MEET

Consider the graphs of a quadratic function and a linear function on the same set of axes.

Notice that we could have:

cutting (2 points of intersection) | **touching** (1 point of intersection) | **missing** (no points of intersection)

If the graphs meet, the coordinates of the points of intersection of the graphs can be found by *solving the two equations simultaneously.*

Example 22 **Self Tutor**

Find the coordinates of the points of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$ without technology.

EXERCISE 1E

1 Without using technology, find the coordinates of the point(s) of intersection of:

a $y = x^2 - 2x + 8$ and $y = x + 6$

b $y = -x^2 + 3x + 9$ and $y = 2x - 3$

c $y = x^2 - 4x + 3$ and $y = 2x - 6$

d $y = -x^2 + 4x - 7$ and $y = 5x - 4$

Example 23 **Self Tutor**

Consider the curves $y = x^2 + 5x + 6$ and $y = 2x^2 + 2x - 4$.

a Solve for x: $x^2 + 5x + 6 = 2x^2 + 2x - 4$.

b Solve for x: $x^2 + 5x + 6 > 2x^2 + 2x - 4$.

Example 24 **Self Tutor**

$y = 2x + k$ is a tangent to $y = 2x^2 - 3x + 4$. Find k.

2 Use a **graphing package** or a **graphics calculator** to find the coordinates of the points of intersection (to 2 decimal places) of the graphs with equations:

a $y = x^2 - 3x + 7$ and $y = x + 5$

b $y = x^2 - 5x + 2$ and $y = x - 7$

c $y = -x^2 - 2x + 4$ and $y = x + 8$

d $y = -x^2 + 4x - 2$ and $y = 5x - 6$

GRAPHING PACKAGE

3 **a** **i** Find where $y = x^2$ meets $y = x + 2$.

ii Solve for x: $x^2 > x + 2$.

b **i** Find where $y = x^2 + 2x - 3$ meets $y = x - 1$.

ii Solve for x: $x^2 + 2x - 3 > x - 1$.

c **i** Find where $y = 2x^2 - x + 3$ meets $y = 2 + x + x^2$.

ii Solve for x: $2x^2 - x + 3 > 2 + x + x^2$.

d **i** Find where $y = \frac{4}{x}$ meets $y = x + 3$.

ii Solve for x: $\frac{4}{x} > x + 3$.

4 For which value of c is the line $y = 3x + c$ a tangent to the parabola with equation $y = x^2 - 5x + 7$?

5 Find the values of m for which the lines $y = mx - 2$ are tangents to the curve with equation $y = x^2 - 4x + 2$.

6 Find the gradients of the lines with y-intercept $(0, 1)$ that are tangents to the curve $y = 3x^2 + 5x + 4$.

7 **a** For what values of c do the lines $y = x + c$ never meet the parabola with equation $y = 2x^2 - 3x - 7$?

b Choose one of the values of c found in part **a** above. Sketch the graphs using technology to illustrate that these curves never meet.

EXERCISE 1E

1 **a** $(1, 7)$ and $(2, 8)$ **b** $(4, 5)$ and $(-3, -9)$

c $(3, 0)$ (touching) **d** graphs do not meet

2 **a** $(0.586, 5.59)$ and $(3.41, 8.41)$ **b** $(3, -4)$ touching

c graphs do not meet **d** $(-2.56, -18.8)$ and $(1.56, 1.81)$

3 **a** **i** $(-1, 1)$ and $(2, 4)$ **ii** $x < -1$ or $x > 2$

b **i** $(-2, -3)$ and $(1, 0)$ **ii** $x < -2$ or $x > 1$

c **i** $(1, 4)$ **ii** $x \in \mathbb{R}$, $x \neq 1$

d **i** $(-4, -1)$ and $(1, 4)$ **ii** $x < -4$ or $x > 1$

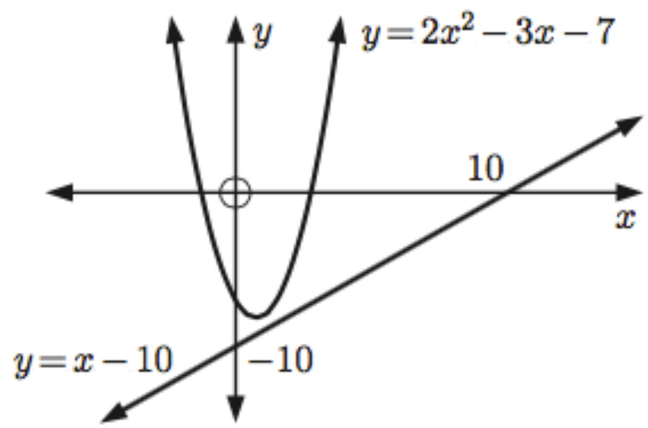
4 $c = -9$

5 $m = 0$ or -8

6 -1 or 11

7 **a** $c < -9$

b example: $c = -10$



F PROBLEM SOLVING WITH QUADRATICS

Some real world problems can be solved using a quadratic equation. We are generally only interested in any **real solutions** which result.

Any answer we obtain must be checked to see if it is reasonable. For example:

- if we are finding a length then it must be positive and we reject any negative solutions
- if we are finding 'how many people are present' then clearly the answer must be an integer.

We employ the following general problem solving method:

Step 1: If the information is given in words, translate it into algebra using a variable such as x for the unknown. Write down the resulting equation. Be sure to define what the variable x represents, and include units if appropriate.

Step 2: Solve the equation by a suitable method.

Step 3: Examine the solutions carefully to see if they are acceptable.

Step 4: Give your answer in a sentence.

Example 25 **Self Tutor**

A rectangle has length 3 cm longer than its width. Its area is 42 cm^2. Find its width.

Example 26 **Self Tutor**

Is it possible to bend a 12 cm length of wire to form the perpendicular sides of a right angled triangle with area 20 cm^2?

EXERCISE 1F

1 Two integers differ by 12 and the sum of their squares is 74. Find the integers.

2 The sum of a number and its reciprocal is $5\frac{1}{5}$. Find the number.

3 The sum of a natural number and its square is 210. Find the number.

4 The product of two consecutive even numbers is 360. Find the numbers.

5 The product of two consecutive odd numbers is 255. Find the numbers.

6 The number of diagonals of an n-sided polygon is given by the formula $D = \frac{n}{2}(n-3)$.

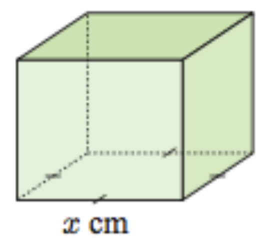
A polygon has 90 diagonals. How many sides does it have?

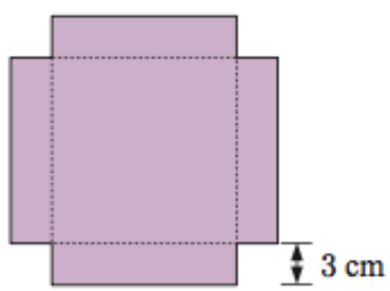
7 The length of a rectangle is 4 cm longer than its width. The rectangle has area 26 cm^2. Find its width.

8 A rectangular box has a square base with sides of length x cm. Its height is 1 cm longer than its base side length. The total surface area of the box is 240 cm^2.

a Show that the total surface area is given by $A = 6x^2 + 4x$ cm^2.

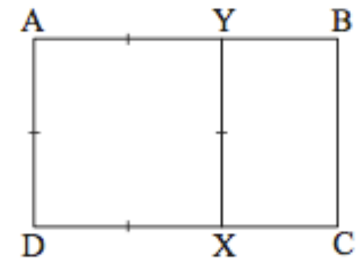
b Find the dimensions of the box.



9 

An open box can hold 80 cm^3. It is made from a square piece of tinplate with 3 cm squares cut from each of its 4 corners. Find the dimensions of the original piece of tinplate.

11 The rectangle ABCD is divided into a square and a smaller rectangle by [XY] which is parallel to its shorter sides.
The smaller rectangle BCXY is *similar* to the original rectangle, so rectangle ABCD is a **golden rectangle**.



The ratio $\frac{AB}{AD}$ is called the **golden ratio**.

Show that the golden ratio is $\frac{1+\sqrt{5}}{2}$.

Hint: Let $AB = x$ units and $AD = 1$ unit.

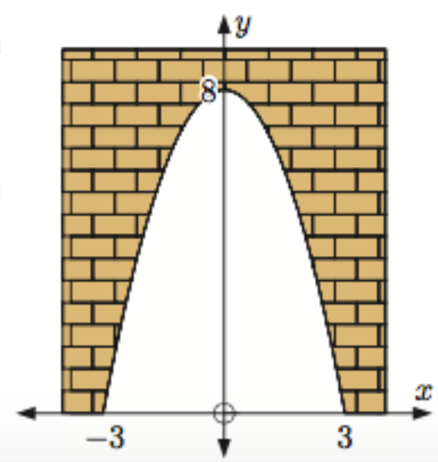
12 Two trains travel a 160 km track each day. The express travels 10 $km\,h^{-1}$ faster and takes 30 minutes less time than the normal train. Find the speed of the express.

13

A group of elderly citizens chartered a bus for \$160. However, at the last minute 8 of them fell ill and had to miss the trip.
As a consequence, the other citizens had to pay an extra \$1 each. How many elderly citizens went on the trip?

14 Answer the **Opening Problem** on page **18**.

15 A truck carrying a wide load needs to pass through the parabolic tunnel shown. The units are metres.
The truck is 5 m high and 4 m wide.



- **a** Find the quadratic function which describes the shape of the tunnel.
- **b** Determine whether the truck will fit.

EXERCISE 1F

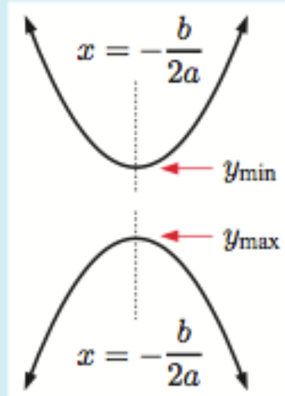
1 7 and −5 or −7 and 5
2 5 or $\frac{1}{5}$
3 14
4 18 and 20 or −18 and −20
5 15 and 17 or −15 and −17
6 15 sides
7 3.48 cm
8 **b** 6 cm by 6 cm by 7 cm
9 11.2 cm square
10 no
12 $61.8\ \text{km}\,\text{h}^{-1}$
13 32
14 **b** The graph is a parabola.
c 21.25 m
d $f(x) = -0.05x^2 + 2x + 1.25$
e yes
15 **a** $y = -\frac{8}{9}x^2 - 8$
b No, as the tunnel is only 4.44 m high when it is the same width as the truck.

G QUADRATIC OPTIMISATION

The process of finding the maximum or minimum value of a function is called **optimisation**.

For the quadratic function $y = ax^2 + bx + c$, we have already seen that the vertex has x-coordinate $-\frac{b}{2a}$.

- If $a > 0$, the **minimum** value of y occurs at $x = -\frac{b}{2a}$
- If $a < 0$, the **maximum** value of y occurs at $x = -\frac{b}{2a}$.



Example 27 **Self Tutor**

Find the maximum or minimum value of the following quadratic functions, and the corresponding value of x:

a $y = x^2 + x - 3$

b $y = 3 + 3x - 2x^2$

EXERCISE 1G

1 Find the maximum or minimum values of the following quadratic functions, and the corresponding values of x:

a $y = x^2 - 2x$

b $y = 7 - 2x - x^2$

c $y = 8 + 2x - 3x^2$

d $y = 2x^2 + x - 1$

e $y = 4x^2 - x + 5$

f $y = 7x - 2x^2$

2 The profit in manufacturing x refrigerators per day, is given by the profit relation $P = -3x^2 + 240x - 800$ dollars.

a How many refrigerators should be made each day to maximise the total profit?

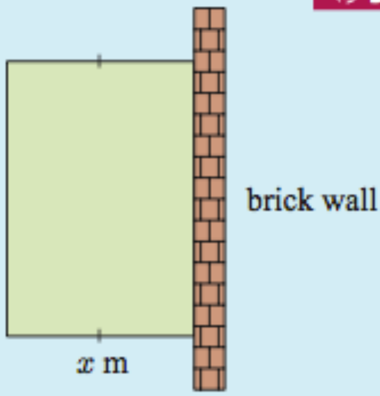
b What is the maximum profit?

Example 28 **Self Tutor**

A gardener has 40 m of fencing to enclose a rectangular garden plot, where one side is an existing brick wall. Suppose the two new equal sides are x m long.

a Show that the area enclosed is given by $A = x(40 - 2x)$ m^2.

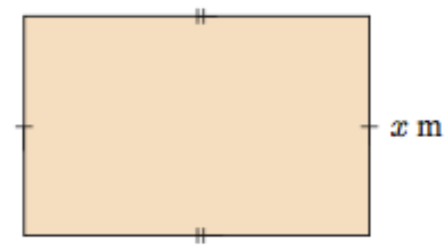
b Find the dimensions of the garden of maximum area.



3 A rectangular plot is enclosed by 200 m of fencing and has an area of A square metres. Show that:

a $A = 100x - x^2$ where x m is the length of one of its sides

b the area is maximised if the rectangle is a square.



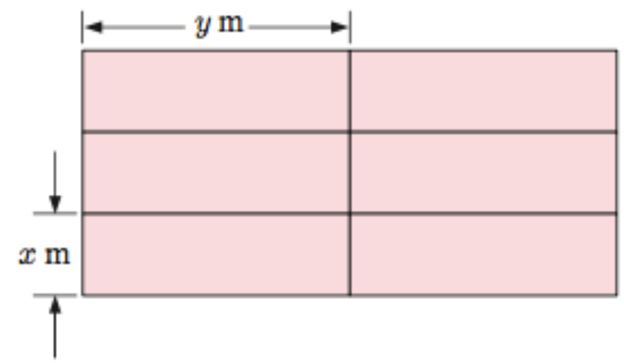
4 Three sides of a rectangular paddock are to be fenced, the fourth side being an existing straight water drain. If 1000 m of fencing is available, what dimensions should be used for the paddock so that it encloses the maximum possible area?

5 1800 m of fencing is available to fence six identical pens as shown in the diagram.

a Explain why $9x + 8y = 1800$.

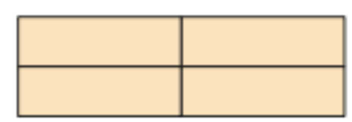
b Show that the area of each pen is given by $A = -\frac{9}{8}x^2 + 225x$ m^2.

c If the area enclosed is to be maximised, what are the dimensions of each pen?

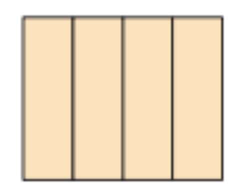


6 500 m of fencing is available to make 4 rectangular pens of identical shape. Find the dimensions that maximise the area of each pen if the plan is:

a



b



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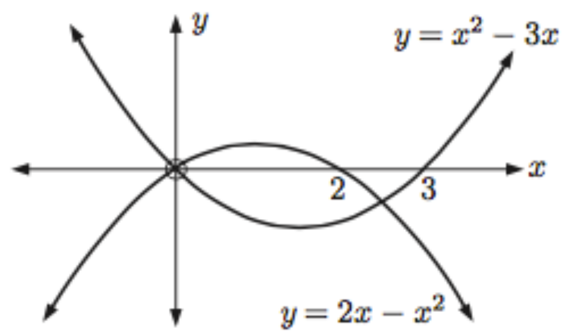
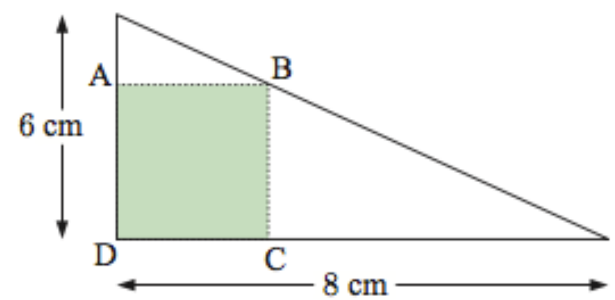


The graphs of $y = x^2 - 3x$ and $y = 2x - x^2$ are illustrated.

a Without using technology, show that the graphs meet where $x = 0$ and $x = 2\frac{1}{2}$.

b Find the maximum vertical separation between the curves for $0 \leqslant x \leqslant 2\frac{1}{2}$.

8 Infinitely many rectangles may be inscribed within the right angled triangle shown alongside. One of them is illustrated.



a Let $AB = x$ cm and $BC = y$ cm.
Use similar triangles to find y in terms of x.

b Find the dimensions of rectangle ABCD of maximum area.

EXERCISE 1G

1 a min. -1, when $x=1$ b max. 8, when $x=-1$

c max. $8\frac{1}{3}$, when $x=\frac{1}{3}$ d min. $-1\frac{1}{8}$, when $x=-\frac{1}{4}$

e min. $4\frac{15}{16}$, when $x=\frac{1}{8}$ f max. $6\frac{1}{8}$, when $x=\frac{7}{4}$

2 a 40 refrigerators b $4000

4 500 m by 250 m 5 c 100 m by 112.5 m

6 a $41\frac{2}{3}$ m by $41\frac{2}{3}$ m b 50 m by $31\frac{1}{4}$ m

7 b $3\frac{1}{8}$ units 8 a $y=6-\frac{3}{4}x$ b 3 cm by 4 cm